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FAST TRACK COMMUNICATION

On the conditions for discrimination between quantum states with minimum error

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Abstract

We provide a simple proof for the necessity of conditions for discriminating with minimum error between a known set of quantum states.

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In quantum communications a transmitting party, Alice, selects from among a set of agreed quantum states to prepare a quantum system for transmission to the receiving party, Bob. Both the set of possible states, $\{\hat{\rho}_i\}$, and the associated probabilities for selection, $\{p_i\}$, are known to Bob but not, of course, the selected state. His task is to determine as well as he can which state was prepared and he does this by choosing a measurement to perform. If the states are not mutually orthogonal then there is no measurement that will reveal the selected state with certainty. The strategy he chooses will depend on the use for which the information is intended and there exist many figures of merit for Bob's measurement [1, 2]. Among these the simplest is the minimum probability of error or, equivalently, the maximum probability for correctly identifying the state. Necessary and sufficient conditions for realizing a minimum error measurement are known [3–6], but it has proven to be easier to prove sufficiency than necessity. This letter presents an appealingly simple proof that the conditions are necessary.

A minimum error measurement will, in general, be a generalized measurement described not by projectors but rather by a probability operator measure (POM) [5], also referred to as a positive operator valued measure [7]. The probability that a generalized measurement gives the result j is

$$P(j) = \mathrm{Tr}(\hat{\rho}\hat{\pi}_j), \tag{1}$$

where $\hat{\pi}_j$ is a probability operator. These are defined, mathematically, by the requirements that

$$\hat{\pi}_i^\dagger = \hat{\pi}_i \tag{2}$$

$$\hat{\pi}_i \geqslant 0 \quad \text{or} \quad \langle\psi|\hat{\pi}_i|\psi\rangle \geqslant 0 \quad \forall\,|\psi\rangle \tag{3}$$

$$\sum_i \hat{\pi}_i = \hat{1}. \tag{4}$$

A minimum error measurement identifies the outcome i with the prepared state $\hat{\rho}_i$ and the probability for correctly identifying the state is therefore

$$P_{\text{corr}} = \sum_i p_i \text{Tr}(\hat{\rho}_i\hat{\pi}_i), \tag{5}$$

and the error probability is, of course, $P_{\text{err}} = 1 - P_{\text{corr}}$.

The conditions for minimum error are

$$\hat{\pi}_j(p_j\hat{\rho}_j - p_k\hat{\rho}_k)\hat{\pi}_k = 0 \quad \forall j, k \tag{6}$$

$$\sum_i p_i\hat{\rho}_i\hat{\pi}_i - p_j\hat{\rho}_j \geqslant 0 \quad \forall j. \tag{7}$$

The latter condition further requires that the operator

$$\hat{\Gamma} = \sum_i p_i\hat{\rho}_i\hat{\pi}_i \tag{8}$$

should be Hermitian, for if it has an anti-Hermitian part then its expectation value can be complex rather than the required real and positive value. It is straightforward to show that the condition (7) is sufficient to minimize the error. To see this let us consider another (primed) measurement associated with the POM $\{\hat{\pi}'_j\}$. The difference between the probabilities for correctly identifying the state with the minimum error and primed measurements is

$$\begin{aligned} P_{\text{corr}} - P'_{\text{corr}} &= \sum_i p_i\text{Tr}(\hat{\rho}_i\hat{\pi}_i) - \sum_j p_j\text{Tr}(\hat{\rho}_j\hat{\pi}'_j) \\ &= \sum_j \text{Tr}[(\hat{\Gamma} - p_j\hat{\rho}_j)\hat{\pi}'_j] \\ &\geqslant 0, \end{aligned} \tag{9}$$

where we have used the completeness condition (4) for the primed probability operators, and the final inequality follows from the assumption that the original (unprimed) measurement minimizes the error probability. The probability operators $\hat{\pi}'_j$ are positive by virtue of the fact that they represent a measurement. If the operators $\hat{\Gamma} - p_j\hat{\rho}_j$ are also positive then it follows immediately that $\text{Tr}\,[(\hat{\Gamma} - p_j\hat{\rho}_j)\hat{\pi}'_j] \geqslant 0$. If we can find a POM that satisfies the inequalities (7) then it will be a minimum error strategy. This establishes the sufficiency of the condition (7).

In order to prove that (7) is also necessary we introduce the manifestly Hermitian operators

$$\hat{G}_j = \frac{1}{2}\sum_i p_i(\hat{\rho}_i\hat{\pi}_i + \hat{\pi}_i\hat{\rho}_i) - p_j\hat{\rho}_j, \tag{10}$$

where the operators $\{\hat{\pi}_i\}$ comprise a minimum error measurement. It is straightforward to show that each of the operators $\hat{G}_j$ must be positive by considering the effects of a single negative eigenvalue. Let us suppose that for one state, $\hat{\rho}_1$, the operator $\hat{G}_1$ has a single negative eigenvalue, $-\lambda$:

$$\hat{G}_1|\lambda\rangle = -\lambda|\lambda\rangle. \tag{11}$$

If this single negative eigenvalue means that there exists a POM with a lower error probability then it *necessarily* follows that the positivity of $\hat{G}_1$ (and by extension of all of the operators $\hat{G}_j$) is a necessary condition for a minimum error POM.

Consider a measurement with probability operators $\hat{\pi}_i'$ related to the operators $\hat{\pi}_i$ by

$$\hat{\pi}_i' = (\hat{1} - \varepsilon|\lambda\rangle\langle\lambda|)\hat{\pi}_i(\hat{1} - \varepsilon|\lambda\rangle\langle\lambda|) + \varepsilon(2-\varepsilon)|\lambda\rangle\langle\lambda|\delta_{i1}, \quad (12)$$

where the positive quantity $\varepsilon \ll 1$. It is easily verified that the set of these primed operators satisfies the conditions (2)–(4) and so represents a valid measurement. The probability that the primed measurement will correctly identify the state is

$$\begin{aligned} P_{\rm corr}' &= \sum_i p_i \mathrm{Tr}(\hat{\rho}_i\hat{\pi}_i') \\ &= P_{\rm corr} - \varepsilon\sum_i \langle\lambda|(\hat{\rho}_i\hat{\pi}_i + \hat{\pi}_i\hat{\rho}_i)|\lambda\rangle + 2\varepsilon p_1\langle\lambda|\hat{\rho}_1|\lambda\rangle + O(\varepsilon^2) \\ &= P_{\rm corr} + 2\varepsilon\lambda + O(\varepsilon^2), \end{aligned} \quad (13)$$

where we have used the eigenvalue property (11). This is clearly greater than $P_{\rm corr}$ and so is at odds with the assumption that $P_{\rm corr}$ is the maximum probability for correctly identifying the state. It follows that the positivity of the operators $\hat{G}_j$ is a *necessary* condition for maximizing the probability of correctly identifying the state or, equivalently, for minimizing the probability of error.

We complete our proof of the necessity of the positivity condition (7) by showing that the operator $\hat{\Gamma}$ must be Hermitian so that

$$\hat{G}_j = \hat{\Gamma} - p_j\hat{\rho}_j. \quad (14)$$

To see this we need only note that

$$\sum_j \mathrm{Tr}(\hat{G}_j\hat{\pi}_j) = 0. \quad (15)$$

Because both $\hat{G}_j$ and $\hat{\pi}_j$ are positive operators, it must then be the case that $\hat{G}_j\hat{\pi}_j = 0$, as may easily be verified by evaluating the trace in the eigenbasis of either operator. Summing this over all j then gives

$$\frac{1}{2}\sum_i p_i(\hat{\pi}_i\hat{\rho}_i - \hat{\rho}_i\hat{\pi}_i) = \frac{1}{2}(\hat{\Gamma}^\dagger - \hat{\Gamma}) = 0, \quad (16)$$

so that the operator $\hat{\Gamma}$ is necessarily Hermitian. This concludes the proof of the necessity of the positivity condition (7) for any minimum error measurement.

We conclude by showing how the equality condition (6) follows from the inequality condition (7). The positivity of the operators $\hat{\Gamma} - p_j\hat{\rho}_j$ and $\hat{\pi}_j$ together with the trivial condition

$$\sum_j \mathrm{Tr}[(\hat{\Gamma} - p_j\hat{\rho}_j)\hat{\pi}_j] = 0 \quad (17)$$

means that

$$(\hat{\Gamma} - p_k\hat{\rho}_k)\hat{\pi}_k = 0 \quad (18)$$

$$\hat{\pi}_j(\hat{\Gamma} - p_j\hat{\rho}_j) = 0. \quad (19)$$

If we premultiply (18) by $\hat{\pi}_j$, postmultiply (19) by $\hat{\pi}_k$ and take the difference then we recover the condition (6). We conclude that together the minimum error conditions (6) and (7) are both sufficient and necessary. For any set of states and preparation probabilities there will exist at least one minimum error measurement with probability operators satisfying these conditions.

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